A CHARACTERISTIC-FREE PROOF OF A BASIC RESULT ON $\mathcal{D}$-MODULES

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Abstract. Let k be a field, let R be a ring of polynomials in a finite number of variables over k, let $\mathcal{D}$ be the ring of k-linear differential operators of R and let $f \in R$ be a non-zero element. It is well-known that R_f, with its natural $\mathcal{D}$-module structure, has finite length in the category of $\mathcal{D}$-modules. We give a characteristic-free proof of this fact. To the best of our knowledge this is the first characteristic-free proof.

1. Introduction.

Throughout this paper k is a field, $R = k[x_1, \dots, x_n]$ is the ring of polynomials in a finite number of variables over k and $\mathcal{D}$ is the ring of k-linear differential operators of R. The natural $\mathcal{D}$-action on R induces a $\mathcal{D}$-module structure on R_f for every $0 \neq f \in R$. The goal of this paper is to give a characteristic-free proof of the following well-known fact.

Theorem 1.1. *R_f has finite length in the category of $\mathcal{D}$-modules.*

In characteristic 0 this is due to J. Bernstein [2, 3] and in characteristic $p > 0$ to R. Bøgvad [5]. In both cases proofs are based on suitable notions of holonomicity but the definitions of holonomicity in each of these two cases are completely different.

Our characteristic-free proof is made possible by V. Bavula's wonderful paper [1] where a characteristic-free definition of holonomic modules is given. But the focus of [1] is the characteristic $p > 0$ case and this assumption is routinely made in the statements and used in the proofs.

In this paper we simplify and characteristic-freeify those of Bavula's results that are needed for a proof of Theorem 1.1.

Finiteness properties of local cohomology modules for regular rings containing a field had originally been proven by two completely different methods in characteristic $p > 0$ [6] and in characteristic 0 [7]. In [9] we used $\mathcal{D}$-modules to give proofs of these finiteness properties that are characteristic-free modulo the fact that $\mathcal{R}_f$, where $\mathcal{R} = k[[x_1, \dots, x_n]]$ is the ring of formal power series in a finite number of variables over k, has finite length in the

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category of k-linear $\mathcal{D}$-modules of $\mathcal{R}$. The proofs of this complete local analogue of Theorem 1.1 are still completely different in characteristic 0 [4] and in characteristic $p > 0$ [8].

Our proof of Theorem 1.1 leads to a characteristic-free proof of the finiteness properties of local cohomology modules over polynomial rings. And it suggests a way to find a similar proof in general, i.e. for all regular local rings containing a field: through a suitable characteristic-free definition of holonomicity in the complete local case that would lead to a proof of an analogue of Theorem 1.1 in this case. Such a definition is yet to be discovered.

This paper is self-contained.

2. Preliminaries.

Let $D_{t,i} = \frac{1}{t!}\frac{\partial^t}{\partial x_i^t} : R \to R$ be the $k[x_1 \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$-linear map that sends x_i^v to $\binom{v}{t}x^{t-v}$ ($D_{0,i}$ is the identity map). Even though $\frac{1}{t!}$ is part of the notation, $D_{t,i}$ exists in all characterisitcs because $\binom{v}{t}$ is an integer.

The ring R is in a natural way a subring of $\mathrm{End}_k R$ (every element of R corresponds to the multiplication by that element on R) and the following equality holds in $\mathrm{End}_k R$.

Proposition 2.1. *$D_{t,i} \cdot f = \Sigma_{s=0}^{s=t} D_{s,i}(f) \cdot D_{t-s,i}$ for every $f \in R$.*

Proof. We have to show that for every $g \in R$

$$D_{t,i}(f \cdot g) = \Sigma_{s=0}^{s=t} D_{s,i}(f) \cdot D_{t-s,i}(g)$$

which is the well-known formula for the higher derivative of a product

$$\frac{\partial^t}{\partial x_i^t}(f \cdot g) = \Sigma_{s=0}^{s=n} \binom{t}{s} \frac{\partial^s}{\partial x_i^s}(f) \cdot \frac{\partial^{t-s}}{\partial x_i^{t-s}}(g)$$

divided by $t!$ □

Corollary 2.2. *(a) $D_{t,i}$ commutes with x_j for $j \neq i$ and with all $D_{s,j}$.*
(b) $D_{t,i}x_i^w = \Sigma_{s=0}^{s=t}\binom{w}{s}x_i^\ell D_{t-s,i}$.
(c) $D_{t,i} \cdot D_{s,j} = \binom{s+t}{s}D_{t+s,i}$.

Proof. (a) and (c) are straightforward and (b) is 2.1 with $f = x_i^w$. □

The ring $\mathcal{D}$ of k-linear differential operators of R is the k-subalgebra of $\mathrm{End}_k R$ generated by R and all the $D_{t,i}$s. Corollary 2.2 implies that the products $\{x_1^{i_1} \cdots x_n^{i_n} \cdot D_{t_1,1} \cdots D_{t_n,n}\}$ where $i_1, \ldots, i_n, t_1, \ldots, t_n$ range over all the $2n$-tuples of non-negative integers, are a k-basis of $\mathcal{D}$. Indeed, every element of $\mathcal{D}$ is by definition a linear combination of products of $D_{t,i}$s and x_js. Using relations 2.2(a)-(c) we can write every such product as a linear combination of products of the form $x_1^{i_1} \cdots x_n^{i_n} \cdot D_{t_1,1} \cdots D_{t_n,n}$. Thus $\mathcal{D}$ is free left R-module on the products $D_{t_1,1} \cdots D_{t_n,n}$ and similarly, it is a free right R-module on these same products.

Corollary 2.3. *$x^w \cdot D_{t,i} \in \mathcal{D}x_i$ if $w > t$ and $x^t \cdot D_{t,i} - (-1)^t \in \mathcal{D}x_i$.*

Proof. Isolating $x_i^w \cdot D_{t,i}$ from 2.2b we get

$$x_i^w \cdot D_{t,i} = D_{t,i} \cdot x_i^w - \Sigma_{s=1}^{s=t} x_i^{w-s} \cdot D_{t-s,i}$$

which implies both containments by induction on t. $\square$

Proposition 2.4. *Let $\mathfrak{m} \subset R$ be a k-rational maximal ideal of R (this means that the natural map $k \hookrightarrow R/\mathfrak{m}$ is bijective). If $\delta \in \mathcal{D}$, we denote $\bar{\delta} \in \mathcal{D}/\mathcal{D}\mathfrak{m}$ the image of δ under the natural surjection $\mathcal{D} \to \mathcal{D}/\mathcal{D}\mathfrak{m}$.*

(i) $\mathcal{D}/\mathcal{D}\mathfrak{m}$ is the k-vector space with basis $\{\overline{D_{t_1,1} \cdots D_{t_n,n}}\}$ as $t_1, \ldots, t_n$ range over all non-negative integers.

(ii) Every element of $\mathcal{D}/\mathcal{D}\mathfrak{m}$ is annihilated by a power of $\mathfrak{m}$ and the socle of $\mathcal{D}/\mathcal{D}\mathfrak{m}$ is generated by $\bar{1}$.

Proof. (i) follows from the fact that $\mathcal{D}$ is a free right R-module on the products $D_{t_1,1} \cdots D_{t_n,n}$ and $\mathcal{D}/\mathcal{D}\mathfrak{m} = \mathcal{D} \otimes_R (R/\mathfrak{m})$.

(ii) Since the natural map $k \hookrightarrow R/\mathfrak{m}$ is bijective, $\mathfrak{m} = (x_1 - c_1, \ldots, x_n - c_n)$ where $c_1, \ldots, c_n \in K$. Viewing $x_j - c_j$ as a new x_j we can assume that $\mathfrak{m} = (x_1, \ldots, x_n)$. According to 2.3, $x_i^{t_i+1}$ annihilates $\{\overline{D_{t_1,1} \cdots D_{t_n,n}}\}$, hence every element of $\mathcal{D}/\mathcal{D}\mathfrak{m}$ is annihilated by a power of $\mathfrak{m}$. Clearly $\bar{1}$ belongs to the socle. It remains to show that every non-zero element z can be sent to the socle by multiplication by an element of R. According to (i) z is a k-linear combination of a finite number of $\overline{D_{t_1,1} \cdots D_{t_n,n}}$. Let $D_{t_1,1} \cdots D_{t_n,n}$ have a maximal $t_1 + \cdots + t_n$ among all the $\overline{D_{t_1,1} \cdots D_{t_n,n}}$ with non-zero coefficients in this linear combination. Hence for every other $\overline{D_{t'_1,1} \cdots D_{t'_n,n}}$ with non-zero coefficient in the linear combination there is j such that $t_j > t'_j$. It follows from 2.2 that $x_j^{t_j} D_{t',j} \in \mathcal{D}\mathfrak{m}$. Hence $x_1^{t_1} \cdots x_n^{t_n} \overline{D_{t'_1,1} \cdots D_{t'_n,n}} = 0$. It similarly follows from 2.2 that $x_1^{t_1} \cdots x_n^{t_n} \overline{D_{t_1,1} \cdots D_{t_n,n}} = (-1)^{t_1+\cdots+t_n} \bar{1}$. Hence $(-1)^{t_1+\cdots+t_n} x_1^{t_1} \cdots x_n^{t_n} z = \bar{1}$. $\square$

Corollary 2.5. *Let $\mathfrak{m} \subset R$ be a maximal ideal such that $R/\mathfrak{m}$ is a finite separable field extension of k. Let M be a $\mathcal{D}$-module and let $z \in M$ be a non-zero element such that its annihilator in R is $\mathfrak{m}$. The set $\{D_{t_1,1} \cdots D_{t_n,n} z\}$, as $t_1, \ldots, t_n$ range over all non-negative integers, is linearly independent over k.*

Proof. Replacing M by the $\mathcal{D}$-submodule generated by z we can assume that M is generated by z. Let K denote the algebraic closure of k, let $R' = K \otimes_k R = K[x_1, \ldots, x_n]$, $\mathcal{D}' = K \otimes_k \mathcal{D}$ and $M' = K \otimes_k M$. Then $\mathcal{D}'$ is the ring of K-linear differential operators of R' and M' is naturally a $\mathcal{D}'$-module. Identifying M with the subset $1 \otimes_k M$ of M' we conclude that it is enough to show that the set $\{D_{t_1,1} \cdots D_{t_n,n} z\} \subset M'$ is linearly independent over K.

Let $\mathfrak{m}_1 \ldots, \mathfrak{m}_s$ be the maximal ideals of R' that lie over $\mathfrak{m}$. Since the field extension $k \hookrightarrow R/\mathfrak{m}$ is separable, $K \otimes_k R/\mathfrak{m}$ is reduced. Therefore $K \otimes_k R/\mathfrak{m} = R'/(\cap_i \mathfrak{m}_i)$. This implies that $R'z = K \otimes_k Rz \cong R'/(\cap_i \mathfrak{m}_i)$

since $Rz \cong R/\mathfrak{m}$. Now M' being generated by z is a surjective image of $\mathcal{D}'/\mathcal{D}'(\cap_i \mathfrak{m}_i)$ via the surjection is $\mathcal{D}'/\mathcal{D}'(\cap_i \mathfrak{m}_i) \overset{\bar{1} \mapsto z}{\to} M'$.

But $\mathcal{D}'/\mathcal{D}'(\cap_i \mathfrak{m}_i) = \mathcal{D}' \otimes_{R'} R'/(\cap_i \mathfrak{m}_i) = \mathcal{D}' \otimes_{R'} (\oplus_i R'/\mathfrak{m}_i) = \oplus_i D'/D'\mathfrak{m}_i$. According to 2.4, the socle of each $D'/D'\mathfrak{m}_i$ is generated by $\bar{1}$, hence so is the socle of $\mathcal{D}'/\mathcal{D}'(\cap_i \mathfrak{m}_i)$. This means the surjection induces a bijection on the socles and therefore it is itself a bijection. Thus $\mathcal{D}'/\mathcal{D}'(\cap_i \mathfrak{m}_i)$ is isomorphic to M via an isomorphism that sends $\overline{D_{t_1,1} \cdots D_{t_n,n}}$ to $\{D_{t_1,1} \cdots D_{t_n,n} z\}$. But the set $\{\overline{D_{t_1,1} \cdots D_{t_n,n}}\}$ is linearly independent (this is a consequence of 2.4 after a natural projection onto some $D'/D'\mathfrak{m}_i$). $\square$

Definition 2.6. *The Bernstein filtration $k = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ on $\mathcal{D}$ is defined by setting $\mathcal{F}_s$ to be the k-linear span of the set of products $\{x_1^{i_1} \cdots x_n^{i_n} \cdot D_{t_1,1} \cdots D_{t_n,n} | i_1 + \dots i_n + t_1 + \cdots + t_n \le s\}$.*

It follows from 2.2 that $\mathcal{F}_i \cdot \mathcal{F}_j \subset \mathcal{F}_{i+j}$.

3. Proof of Theorem 1.1

The technical heart of our proof is the following proposition.

Proposition 3.1. *(cf.* [1, 9.3]*) Assume the field k is separable. Let M be a $\mathcal{D}$-module and let $z \in M$ be an element such that the annihilator of z in R is a prime ideal of R. Then* $\dim_k(\mathcal{F}_i z) \ge \binom{n+i}{i}$.

Proof. Let $d = \dim R/P$, let $h = n - d$, and let K be the fraction field of R/P. Since the transcendence degree of K over k equals d and k is separable, after a possible permutation of indices we can assume that $x_{h+1}, \dots, x_n$ are algebraically independent over k in K and K is finite and separable over the field of rational functions $\mathcal{K} = k(x_{h+1}, \dots, x_n)$.

Let $R' = \mathcal{K} \otimes_R R = \mathcal{K}[x_1, \dots, x_h]$, let $\mathcal{D}'$ be the ring of $\mathcal{K}$-linear differential operators of R' and let $M' = \mathcal{K} \otimes_R M$. The ring $\mathcal{D}'$ is a free R'-module on the products $D_{t_1,1} \cdots D_{t_h,h}$. Since each such product commutes with x_j for $j > h$, its action on M naturally extends to an action on M' making M' a $\mathcal{D}'$-module. It follows from 2.5 that the set of elements $\{D_{t_1,1} \cdots D_{t_h,h} z\} \subset M'$, as $t_1, \dots, t_h$ run through all non-negative integers, is linearly independent over $\mathcal{K}$. Setting $R'' = k[x_{h+1}, \dots, x_n]$ ($\mathcal{K}$ is the fraction field of R'') we conclude that the sum $\Sigma_{t_1,\dots,t_h} R'' D_{t_1,1} \cdots D_{t_h,h} z$ of R''-submodules of M is direct, i.e. the natural surjective R''-module map $\oplus_{t_1,\dots,t_h} R' D_{t_1,1} \cdots D_{t_h,h} \to \Sigma_{t_1,\dots,t_h} R'' D_{t_1,1} \cdots D_{t_h,h} z$ that sends every product $D_{t_1,1} \cdots D_{t_h,h} \in \oplus_{t_1,\dots,t_h} R'' D_{t_1,1} \cdots D_{t_h,h}$ to $D_{t_1,1} \cdots D_{t_h,h} z \in M$ is bijective. And this implies that the set $\{x_{h+1}^{i_{h+1}} \cdots x_n^{i_n} D_{t_1,1} \cdots D_{t_h,h} z\}$ of elements of M, as $t_1, \dots, t_h, i_{h+1}, \dots, i_n$ run over all non-negative integers, is linearly independent over k.

The elements of this set with $t_1 + \cdots t_h + i_1 + \cdots + i_n \le i$ belong to $\mathcal{F}_i z$. The number of these elements equals the number of monomials of degree at most i in n variables which is well-known to equal $\binom{n+i}{i}$. $\square$

Definition 3.2. *A k-filtration on a $\mathcal{D}$-module M is an ascending chain of k-vector spaces $M_0 \subset M_1 \subset M_2 \subset \dots$ such that $\cup_i M_i = M$ and $\mathcal{F}_i M_j \subset M_{i+j}$ for all i and j.*

For example, the Bernstein filtration on $\mathcal{D}$ is a k-filtration.

Corollary 3.3. *Let $M_0 \subset M_1 \subset M_2 \subset \dots$ be a k-filtration on a $\mathcal{D}$-module M. There exists an integer j such that $\dim_k M_i \geq \binom{n+i-j}{i-j}$ for all $i \geq j$.*

Proof. Let k' be the algebraic closure of k and $R' = k' \otimes_k R$. Then the ring of k'-linear differential operators of R' is $\mathcal{D}' = k' \otimes_k \mathcal{D}$ and $M' = k' \otimes_R M$ is in a natural way a $\mathcal{D}'$-module with k'-filtration $M'_0 \subset M'_1 \subset M'_2 \subset \dots$ where $M'_i = k' \otimes_k M_i$. Since $\dim_k M_i = \dim_{k'} M'_i$, we can and do assume that k is algebraically closed and in particular separable.

Let $P \subset R$ be an associated prime ideal of M in R. This means there exists an element $z \in M$ such that the annihilator of z in R is P. Let j be the smallest integer such that $z \in M_j$. Clearly $M_i \supset \mathcal{F}_{i-j} z$, so we are done by 3.1 $\square$

The following definition of holonomicity is equivalent to but somewhat simpler than Bavula's original definition [1, pp. 185, 198]; in particular we do not require the module M to be finitely generated. But Theorem 3.5 implies that every holonomic module is finitely generated (this fact is not used in the sequel).

Definition 3.4. *A $\mathcal{D}$-module M is holonomic if it has a k-filtration with $\dim_k M_i \leq C i^n$ for all i where C is a constant independent of i.*

Theorem 3.5. *(cf.* [1, 9.6]*) Every holonomic $\mathcal{D}$-module has finite length in the category of $\mathcal{D}$-modules. In fact if $M_0 \subset M_1 \subset \dots$ is a k-filtration on M with $\dim_k M_i \leq C i^n$, then the length of M in the category of $\mathcal{D}$-modules is at most $n!C$.*

Proof. Let $0 = M^0 \subset M^1 \subset \dots M^\ell = M$ be a filtration of M in the category of $\mathcal{D}$-modules. Then $(M^s/M^{s-1})_i = (M_i \cap M^s)/(M_i \cap M^{s-1})$ is a k-filtration on the $\mathcal{D}$-module M^s/M^{s-1}. Hence there is an integer j_s such that $\dim_k (M^s/M^{s-1})_i \geq \binom{n+i-j_s}{i-j_s}$ for all $i \geq j_s$.

But $M_i = \oplus_{s=1}^{s=\ell} (M^s/M^{s-1})_i$ because these are vector spaces over a field k. Therefore $\dim_k M_i = \Sigma_{s=1}^{s=\ell} \dim_k (M^s/M^{s-1})_i \geq \Sigma_{s=1}^{s=\ell} \binom{n+i-j_s}{i-j_s}$ for all sufficiently big i. This implies $C i^n \geq \Sigma_{s=1}^{s=\ell} \binom{n+i-j_s}{i-j_s}$ for all sufficiently big i.

But $\binom{n+i-j_s}{i-j_s}$, for fixed n and j_s, is a polynomial in i of degree n and top coefficient $\frac{1}{n!}$. Hence $p(i) = \Sigma_{s=1}^{s=\ell} \binom{n+i-j_s}{i-j_s}$ is a polynomial in i of degree n and top coefficient $\frac{\ell}{n!}$. The inequality $C i^n \geq p(i)$ holds for all sufficiently big i only if $C \geq \frac{\ell}{n!}$, i.e. $\ell \leq n!C$. $\square$

If M is a $\mathcal{D}$-module and $f \in R$ is a non-zero element, the module M_f acquires a structure of $\mathcal{D}$-module as follows. The formula 2.1 implies

$$f \cdot D_{t.i} = D_{t,i} \cdot f - \Sigma_{s=1}^{s=n} D_{s,i}(f) \cdot D_{t-s,i}.$$

Replacing f by f^j in this equality and then applying it to $\frac{m}{f^j} \in M_f$ and multiplying on the left by f^{-j} we get

$$D_{t.i}(\frac{m}{f^j}) = f^{-j} \cdot D_{t,i}(m) - \Sigma_{s=1}^{s=n} f^{-j} \cdot D_{s,i}(f^j) \cdot D_{t-s,i}(\frac{m}{f^j}) \tag{1}$$

This leads to a definition of the action of $D_{t,i}$ on M_f by induction on t the case $t = 0$ being trivial (since $D_{0,i}$ is the identity map).

Modules of type M_f are not considered in [1].

Corollary 3.6. *If M is a holonomic module and $f \in R$, then M_f is holonomic.*

Proof. Let $M_0 \subset M_1 \subset \dots$ be a k-filtration of M with $\dim_k M_i \le Ci^n$. Let d be the degree of f and let $M'_0 \subset M'_1 \subset \dots$ be the filtration on M_f defined by $M'_i = \{\frac{m}{f^i} | m \in M_{i(d+1)}\}$. We claim this is a k-filtration, i.e. $\cup_i M'_i = M_f$ and $\mathcal{F}_i M'_j \subset M'_{i+j}$ for all i and j.

Indeed, let $\frac{m}{f^w} \in M_f$ be any element. Assume $m \in M_u$. If $u \le w(d+1)$, then $m \in M_{w(d+1)}$ hence $\frac{m}{f^w} \in M'_w$. If $u > w(d+1)$ let $v = u - w(d+1)$. Since $f^v \in \mathcal{F}_{vd}$, it follows that $f^v \cdot m \in M_{vd+u}$. Since $vd+u = (v+w)(d+1)$, we conclude that $\frac{m}{f^w} = \frac{f^v \cdot m}{f^{w+v}} \in M'_{v+w}$. This shows that $\cup_i M'_i = M_f$.

To prove that $\mathcal{F}_i M'_j \subset M'_{i+j}$ all we need to show is that $x_u M'_j \subset M'_{j+1}$ and $D_{t,u} M'_j \subset M'_{t+j}$ for every $u \in \{1, 2, \dots, n\}$. Let $\frac{m}{f^j} \in M'_j$ where $m \in M_{j(d+1)}$.

Since $x_u \cdot f \in \mathcal{F}_{d+1}$, it follows that $(x_u \cdot f)m \in M'_{(j+1)(d+1)}$. Therefore $x_u \cdot \frac{m}{f^j} = \frac{(x_u \cdot f)m}{f^{j+1}} \in M'_{j+1}$. This shows that $x_u M'_j \subset M'_{j+1}$.

To prove that $D_{t,u} M'_j \subset M'_{t+j}$ we use induction on t the case $t = 0$ being trivial since $D_{0,u}$ is the identity map. It is enough to show that all the terms on the right side of (1), i.e. $f^{-j} \cdot D_{t,u}(m)$ and $f^{-j} \cdot D_{s,u}(f^j) \cdot D_{t-s,u}(\frac{m}{f^j})$, where $s \ge 1$, belong to M'_{t+j}.

Since $f \in \mathcal{F}_d$, $D_{t,u} \in \mathcal{F}_t$ and $m \in M_{j(d+1)}$, it follows that $f^t \cdot D_{t,u} \in \mathcal{F}_{td+t}$ and $f^t \cdot D_{t,u}(m) \in M_{j(d+1)+td+t=(t+j)(d+1)}$. Thus $f^{-j} \cdot D_{t,u}(m) = \frac{f^t \cdot D_{t,u}(m)}{f^{t+j}}$ belongs to M'_{t+j} because the top of this fraction belongs to $M_{(t+j)(d+1)}$.

If $s \ge 1$, then $D_{t-s,u}(\frac{m}{f^j}) \in M'_{t-s+j}$ by the induction hypothesis, i.e. there exists $m_{t-s} \in M_{(t-s+j)(d+1)}$ such that $D_{t-s,u}(\frac{m}{f^j}) = \frac{m_{t-s}}{f^{t-s+j}}$. It follows by induction on j using formula 2.1 that $D_{s,i}(f^j)$ is divisible by f^{j-s}, i.e. $D_{s,i}(f^j) = f^{j-s} \cdot f_s$. Hence $f^{-j} \cdot D_{s,u}(f^j) \cdot D_{t-s,u}(\frac{m}{f^j}) = \frac{f_s \cdot m_{t-s}}{f^{t+j}}$. Since the polynomial $D_{s,i}(f^j)$ has degree $dj - s$, the polynomial f_s has degree $ds - s$. Hence $f_s \cdot m_{t-s} \in M_{ds-s+(t-s+j)(d+1)} \subset M_{(t+j)(d+1)}$. The latter containment is because $ds - s + (t - s + j)(d + 1) \le (t + j)(d + 1)$. This shows that $f^{-j} \cdot D_{s,u}(f^j) \cdot D_{t-s,u}(\frac{m}{f^j}) \in M'_{t+j}$ and completes the proof that $D_{t,u} M'_j \subset M'_{t+j}$ which in turn completes the proof of our claim that $M'_0 \subset M'_1 \subset \dots$ is a k-filtration.

Clearly, $\dim_k M'_i \leq \dim_k M_{i(d+1)} \leq C(i(d+1))^n$. This implies that $\dim_k M'_i \leq C' i^n$ where $C' = C(d+1)^n$. ☐

The filtration by degree on the $\mathcal{D}$-module $M = R$ (i.e. M_i consists of all the polynomials of degree at most i) shows that R is a holonomic $\mathcal{D}$-module. Now Theorem 1.1 follows from 3.6. ☐

4. Some open problems

1. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in the category of $\mathcal{D}$-modules. It is not hard to see that if M is holonomic, then so are M' and M''. In characteristic 0 the converse is also true, i.e. if M' and M'' are holonomic, then so is M. Is this true in characteristic $p > 0$ as well?

2. Let M be a holonomic $\mathcal{D}$-module. Since M has finite length, it is finitely generated as a $\mathcal{D}$-module. This implies that there is a k-filtration $M_0 \subset M_1 \subset \ldots$ on M such that M_0 is finite-dimensional over k and $M_i = \mathcal{F}_i M_0$ (just take M_0 to be the k-span of a finite set of $\mathcal{D}$-generators of M). It is not hard to show that $\limsup_{n\to\infty} \frac{\dim_k M_i}{i^n}$ is independent of M_0. It is well-known that $\lim_{n\to\infty} \frac{\dim_k M_i}{i^n}$ exists in characteristic 0 and, moreover, $n!(\lim_{n\to\infty} \frac{\dim_k M_i}{i^n})$ is an integer in this case (called the multiplicity of M). Is $n!(\limsup_{n\to\infty} \frac{\dim_k M_i}{i^n})$ an integer in characteristic $p > 0$? Does $\lim_{n\to\infty} \frac{\dim_k M_i}{i^n}$ exist in characteristic $p > 0$?

Since these problems are open only in characteristic $p > 0$, it is worth pointing out that Bavula [1] has given some striking examples of properties that hold in characteristic 0 but fail in characteristic $p > 0$. We briefly mention some of them.

Let a $\mathcal{D}$-module M be generated by a finite set $z_1 \ldots, z_s \in M$. Let M_0 be the k-linear span of $z_1, \ldots, z_s$ and let $M_i = \mathcal{F}_i M_0$. Bavula defines the dimension of M as $\inf\{r \in \mathbb{R} | \dim_k M_i < i^r\}$ for all sufficiently big i. It is not hard to show that this definition is independent of a particular choice of a finite set of generators. In characteristic zero it coincides with the usual definition of the dimension of a finitely generated $\mathcal{D}$-module.

Bavula shows [1, 9.4] that $\dim M \geq n$ for every finitely generated $\mathcal{D}$-module M, an analog of the celebrated characteristic zero Bernstein inequality. This inequality is straightforward from 3.1.

Yet Bavula also shows that there are major differences between characteristic zero and characteristic $p > 0$ cases. These are

(a) in characteristic zero the set of possible values of $\dim M$ is all integers between n and $2n$ while in characteristic $p > 0$ it is the set of all real numbers between n and $2n$, and

(b) in characteristic zero a finitely generated $\mathcal{D}$-module M is holonomic if and only if its dimension is n while in characteristic $p > 0$ there exist M such that $\dim M = n$ yet M is not holonomic.

3. Perhaps the most interesting open problem is to find a characteristic-free proof of the fact that $\mathcal{R}_f$ has finite length in the category of k-linear $\mathcal{D}$-modules of the ring $\mathcal{R}$ of formal power series in a finite number of variables over k. Our proof of Theorem 1.1 suggests that a suitable characteristic-free definition of holonomicity could lead to such a proof.

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